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Buckling of a twisted and compressed rod

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Abstract

We consider the problem of determining the stability boundary for an elastic rod under thrust and torsion. The constitutive equations of the rod are such that both shear of the cross-section and compressibility of the rod axis are considered. The stability boundary is determined from the bifurcation points of a single nonlinear second order differential equation that is obtained by using the first integrals of the equilibrium equations. The type of bifurcation is determined for parameter values. It is shown that the bifurcating branch is the branch with minimal energy. Finally, by using the first integral, the solution for one specific dependent variable is expressed in terms of elliptic integrals. The solution pertaining to the complete set of equilibrium equations is obtained by numerical integration.

Keywords: Greenhill problem; Bifurcation

1. Introduction

The problem of determining the stability boundary of a twisted and axially compressed rod is indeed an old one. Suppose that the rod is loaded by a compressive force of intensity P and a torsional couple of intensity M_t. For the case of a rod described by Bernoulli–Euler rod theory, the critical value of the load parameters (P_{cr}, M_{tcr}) was determined by Greenhill in 1883 (see Timoshenko and Gere, 1961). Later the problem was treated by many authors. We mention the works of Biezeno and Grammel (1953), Beck (1955), Kovari (1969), Antman and Kenny (1981), Coleman et al. (1992), Atanackovic and Glavardanov (1996), van der Heijden et al. (1998) and van der Heijden and Thompson (2000). Besides the value of critical load parameters (P_{cr}, M_{tcr}) another important result concerns the *type* of bifurcation at the critical point. It is known that in the case of the Bernoulli–Euler model the rod loaded with a compressive force only exhibits a super-critical bifurcation while the rod loaded by torque only exhibits a subcritical bifurcation. In our previous work Atanackovic and Glavardanov (1996) we determined, analytically, the value of load

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parameters $(P^*_{\mathrm{cr}}, M^*_{\mathrm{tcr}})$ separating sub from the super critical bifurcation. Also in Atanackovic and Glavardanov (1996) the values $(P_{\mathrm{cr}}, M_{\mathrm{tcr}})$ are determined for a rod described by Haringx's type of constitutive equation.

Our intention in this work is to reconsider the problem of stability and bifurcation of a rod, described by Haringx's type of constitutive equation and loaded by a force and a couple. We shall use the Euler (static) method of adjacent equilibrium configuration to analyze stability. First we shall reduce the system of nonlinear equilibrium, geometrical and constitutive equations (altogether 18 equations) to a single nonlinear second order equation. We determine the critical load parameters as well as type of bifurcation from the bifurcation points of this equation. This is a significant reduction, since in Béda et al. (1992) and Atanackovic and Glavardanov (1996) the bifurcation analysis was performed on two second order nonlinear differential equations. Also the reduced system, when linearized, has a null space of dimension one. Thus, we obtain single bifurcation equation for the reduced system. In earlier approaches (see Béda et al., 1992; Atanackovic and Glavardanov, 1996) the null space of dimension two led to two bifurcation equations, one of which was eliminated using symmetry considerations.

Another characteristic of our approach is that the second order equation that we deal with is the Euler–Lagrange equation of an energy type functional and possesses a first integral. We shall use this first integral to express the solution in terms of elliptic integrals. Thus, our results represent a generalization of the results of Iljohin (1979) p. 38 for Bernoulli–Euler rod. On the other hand our results may be viewed as a specialization of the results of Antman and Kenny (1981) to a specific constitutive equation. Due to this specialization we were able to obtain more information about the solution than presented in Antman and Kenny (1981). Namely, we obtained a first integral and representation of the solution in terms of elliptic integrals and we obtained the type of bifurcation for *all* values of load and material parameters.

2. Formulation

Consider an elastic rod O_1O_2 shown in Fig. 1. The rod is naturally straight and loaded by a concentrated compressive force and a couple at its end O_2. The compressive force $\mathbf{P} = -P\mathbf{e}_{10}$ is of constant intensity $P = \text{const.}$ and oriented along the $\bar{x}_{10}$ axis of a fixed rectangular Cartesian coordinate system $\bar{x}_{10}$, $\bar{x}_{20}$ and $\bar{x}_{30}$ with unit vectors $\mathbf{e}_{10}$, $\mathbf{e}_{20}$ and $\mathbf{e}_{30}$, respectively. The couple is given as $\mathbf{T} = M_t\mathbf{e}_{10}$ with $M_t = \text{const.}$ The end O_1 of the rod is fixed to a unmovable rigid plate, lying in the $\bar{x}_{20} - \bar{x}_{30}$ coordinate plane so that the cross-section of the rod that is in contact with the rigid plate does not have any rotation or translation (welded end). For interpretation of this boundary condition as well as for the review of other possibilities of specifying boundary condition for shearable rods (see Antman and Christoforis, 1997). At the end O_2 the rod is welded to a movable rigid plate that can move freely but must remain parallel to the coordinate plane $\bar{x}_{10} = 0$.

Let S be the arc-length of the rod axis in the undeformed state: $S \in [0, L]$ where L is the length of the rod. We specify the configuration of the rod by one vector function $\mathbf{r}(S)$ specifying the position of a point on the rod axis and by orientation of the Cartesian coordinate system with axes $\bar{x}_1, \bar{x}_2, \bar{x}_3$ oriented along the normal to the cross-section and along the principal directions of the rod cross-section at an arbitrary point O of the rod axis, respectively. Thus we have

$$\mathbf{r}(S) = x_{10}\mathbf{e}_{10} + x_{20}\mathbf{e}_{20} + x_{30}\mathbf{e}_{30}. \tag{1}$$

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the unit vectors along the $\bar{x}_1, \bar{x}_2, \bar{x}_3$ respectively. The orientation of the system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ with respect to the unit vectors parallel with $\mathbf{e}_{10}$, $\mathbf{e}_{20}$, $\mathbf{e}_{30}$ and passing through point O is given by three Euler type of angles. We use a set of the 1-3-2 Euler angles (also called ship angles, see Lurie, 1961) that transform $\mathbf{e}_{10}$,

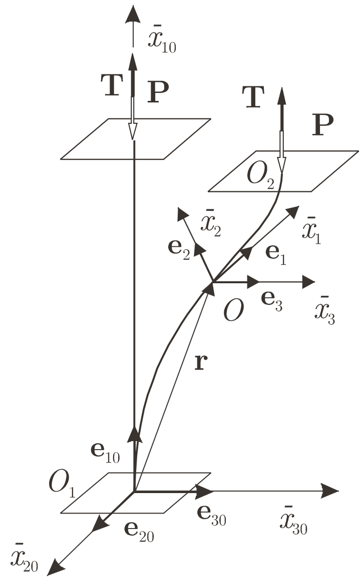

Fig. 1. Coordinate system and load configuration.

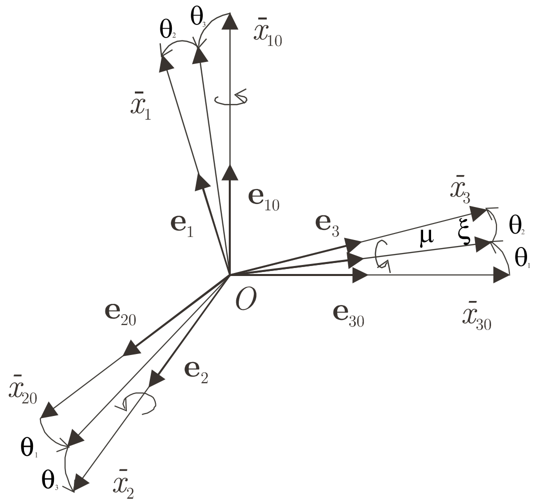

Fig. 2. Euler type of angles defining local coordinate system.

$\mathbf{e}_{20}$, $\mathbf{e}_{30}$ to $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ by the sequence of three rotations. The first is rotation of amount θ_1 about the $\bar{x}_{10}$ axis. The next rotation is about ξ axis for an amount θ_3 (see Fig. 2). The last rotation is of amount θ_2 about the $\bar{x}_2$ axis. All rotations are performed counterclockwise.

The vector $\boldsymbol{\omega}$ (the angular velocity vector) is

$$\boldsymbol{\omega} = \theta_1'\mathbf{e}_{10} + \theta_3'\boldsymbol{\mu} + \theta_2'\mathbf{e}_2, \tag{2}$$

where $(\cdot)' = \mathrm{d}/\mathrm{d}S(\cdot)$ and $\boldsymbol{\mu}$ is the unit vector along the ξ axis. From (2) we obtain the components of $\boldsymbol{\omega}$ in the local coordinate system $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Thus, $\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3$ with

$$\begin{aligned}\omega_1 &= \theta_1' \cos\theta_2 \cos\theta_3 - \theta_3' \sin\theta_2,\\ \omega_2 &= \theta_2' - \theta_1' \sin\theta_3,\\ \omega_3 &= \theta_1' \cos\theta_3 \sin\theta_2 + \theta_3' \cos\theta_2.\end{aligned} \tag{3}$$

Let Γ be a vector defined by

$$\Gamma = \mathbf{r}' - \mathbf{e}_1. \tag{4}$$

In the local coordinate system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ the vector Γ may be expressed as

$$\Gamma = \Gamma_1\mathbf{e}_1 + \Gamma_2\mathbf{e}_2 + \Gamma_3\mathbf{e}_3. \tag{5}$$

Then the six quantities $(\omega_1, \omega_2, \omega_3)$ and $(\Gamma_1, \Gamma_2, \Gamma_2)$ are the *strains* for the rod theory that we use (see Antman, 1995; Atanackovic, 1997).

The equilibrium equations for the rod can be written as

$$\mathbf{F}' = 0, \quad \mathbf{M}' = -\mathbf{r}' \times \mathbf{F}, \tag{6}$$

where $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ is the contact force, $\mathbf{M} = M_1\mathbf{e}_1 + M_2\mathbf{e}_2 + M_3\mathbf{e}_3$ is the contact couple and we assumed that there are no distributed forces or couples. Following Eliseyev (1988) we assume the constitutive equations for the contact force and contact couple in the form

$$\begin{aligned}\mathbf{F} &= B_{11}\Gamma_1\mathbf{e}_1 + B_{22}\Gamma_2\mathbf{e}_2 + B_{33}\Gamma_3\mathbf{e}_3\\ \mathbf{M} &= A_{11}\omega_1\mathbf{e}_1 + A_{22}\omega_2\mathbf{e}_2 + A_{33}\omega_3\mathbf{e}_3,\end{aligned} \tag{7}$$

where $B_{11}, B_{22}, \ldots, A_{33}$ are constants. Note that (4) may be written as

$$\mathbf{r}' = \Gamma + \mathbf{e}_1. \tag{8}$$

In the analysis that follows we assume that the rod has axial symmetry so that $A_{22} = A_{33}$ and $B_{22} = B_{33}$.

Introducing the dimensionless quantities

$$\begin{aligned}&\overline{F}_1 = \frac{F_1L^2}{A_{22}}, \quad \overline{F}_2 = \frac{F_2L^2}{A_{22}}, \quad \overline{F}_3 = \frac{F_3L^2}{A_{22}}, \quad \lambda = \frac{PL^2}{A_{22}},\\ &\overline{M}_1 = \frac{M_1L}{A_{22}}, \quad \overline{M}_2 = \frac{M_2L}{A_{22}}, \quad \overline{M}_3 = \frac{M_3L}{A_{22}}, \quad m = \frac{M_tL}{A_{22}},\\ &\bar{x}_{10} = \frac{x_{10}}{L}, \quad \bar{x}_{20} = \frac{x_{20}}{L}, \quad \bar{x}_{30} = \frac{x_{30}}{L}, \quad t = \frac{S}{L},\\ &\alpha = \frac{A_{22}}{B_{11}L^2}, \quad \beta = \frac{A_{22}}{B_{22}L^2}, \quad \gamma = \alpha - \beta,\end{aligned} \tag{9}$$

and by using (7) in (3), (6) and (8) we obtain

$$
\begin{aligned}
&\dot{\overline{F}}_1 + \overline{F}_3\overline{M}_2 - \overline{F}_2\overline{M}_3 = 0,\\
&\dot{\overline{F}}_2 + \overline{F}_1\overline{M}_3 - \frac{A_{22}}{A_{11}}\overline{F}_3\overline{M}_1 = 0,\\
&\dot{\overline{F}}_3 + \frac{A_{22}}{A_{11}}\overline{F}_2\overline{M}_1 - \overline{F}_1\overline{M}_2 = 0,\\
&\dot{\overline{M}}_1 = 0,\\
&\dot{\overline{M}}_2 + \left(1 - \frac{A_{22}}{A_{11}}\right)\overline{M}_1\overline{M}_3 = \overline{F}_3\left[1 + \gamma\overline{F}_1\right],\\
&\dot{\overline{M}}_3 + \left(\frac{A_{22}}{A_{11}} - 1\right)\overline{M}_1\overline{M}_2 = -\overline{F}_2\left[1 + \gamma\overline{F}_1\right],\\
&\dot{\theta}_1 = \frac{c_2}{c_3}\frac{A_{22}}{A_{11}}\overline{M}_1 + \frac{s_2}{c_3}\overline{M}_3,\\
&\dot{\theta}_2 = \frac{c_2 s_3}{c_3}\frac{A_{22}}{A_{11}}\overline{M}_1 + \overline{M}_2 + \frac{s_2 s_3}{c_3}\overline{M}_3,\\
&\dot{\theta}_3 = -s_2\frac{A_{22}}{A_{11}}\overline{M}_1 + c_2\overline{M}_3,\\
&\dot{\bar{x}}_{10} = c_2 c_3\left(1 + \alpha\overline{F}_1\right) - \beta s_3\overline{F}_2 + \beta c_3 s_2\overline{F}_3,\\
&\dot{\bar{x}}_{20} = (c_1 c_2 s_3 + s_1 s_2)\left(1 + \alpha\overline{F}_1\right) + \beta c_1 c_3\overline{F}_2 + \beta(c_1 s_2 s_3 - c_2 s_1)\overline{F}_3,\\
&\dot{\bar{x}}_{30} = (c_2 s_1 s_3 - c_1 s_2)\left(1 + \alpha\overline{F}_1\right) + \beta s_1 c_3\overline{F}_2 + \beta(s_1 s_2 s_3 + c_1 c_2)\overline{F}_3,
\end{aligned}
\tag{10}
$$

where $c_1 = \cos\theta_1$, $s_1 = \sin\theta_1, \ldots, s_3 = \sin\theta_3$ and we assumed that $c_3 \neq 0$. The boundary conditions corresponding to (10) read

$$
\begin{aligned}
&\overline{F}_1(1) = -\lambda, \quad \overline{F}_2(1) = 0, \quad \overline{F}_3(1) = 0,\\
&\overline{M}_1(1) = m, \quad \theta_2(1) = 0, \quad \theta_3(1) = 0,\\
&\theta_1(0) = 0, \quad \theta_2(0) = 0, \quad \theta_3(0) = 0,\\
&\bar{x}_{10}(0) = 0, \quad \bar{x}_{20}(0) = 0, \quad \bar{x}_{30}(0) = 0.
\end{aligned}
\tag{11}
$$

Let $\overline{F}_1, \overline{F}_2, \overline{F}_3, \ldots, \bar{x}_{30}$ be a solution to (10) and (11). Then an arbitrary rotation about the x_{10} axis leads to another solution, i.e., $\overline{F}_1, \overline{F}_2\cos\delta + \overline{F}_3\sin\delta, -\overline{F}_2\sin\delta + \overline{F}_3\cos\delta, \ldots, -\bar{x}_{20}\sin\delta + \bar{x}_{30}\cos\delta$ is a solution to (10) and (11) for arbitrary δ. Note that (10, line 4) together with (11, line 4) lead to

$$
\overline{M}_1 = m. \tag{12}
$$

For further analysis we shall need the first six equations of the system (10), that after the use of (12) become

$$
\begin{aligned}
&\dot{\overline{F}}_1 + \overline{F}_3\overline{M}_2 - \overline{F}_2\overline{M}_3 = 0,\\
&\dot{\overline{F}}_2 + \overline{F}_1\overline{M}_3 - \frac{A_{22}}{A_{11}}m\overline{F}_3 = 0,\\
&\dot{\overline{F}}_3 + \frac{A_{22}}{A_{11}}m\overline{F}_2 - \overline{F}_1\overline{M}_2 = 0,\\
&\dot{\overline{M}}_2 + \left(1 - \frac{A_{22}}{A_{11}}\right)m\overline{M}_3 = \overline{F}_3\left[1 + \gamma\overline{F}_1\right],\\
&\dot{\overline{M}}_3 + \left(\frac{A_{22}}{A_{11}} - 1\right)m\overline{M}_2 = -\overline{F}_2\left[1 + \gamma\overline{F}_1\right].
\end{aligned}
\tag{13}
$$

It can be seen that the system (13) has the following first integrals

$$\begin{aligned} \overline{F}_1 m + \overline{F}_2\overline{M}_2 + \overline{F}_3\overline{M}_3 &= -\lambda m, \\ \overline{F}_1^2 + \overline{F}_2^2 + \overline{F}_3^2 &= \lambda^2. \end{aligned} \tag{14}$$

We proceed now to simplify (13). Thus, we introduce the variables

$$\begin{aligned} X_2 &= \overline{M}_2^2 + \overline{M}_3^2, \\ X_3 &= \overline{F}_2\overline{M}_2 + \overline{F}_3\overline{M}_3, \\ X_4 &= \overline{F}_2\overline{M}_3 - \overline{F}_3\overline{M}_2. \end{aligned} \tag{15}$$

By differentiating (14, line 1) and by using (13) we find that

$$\dot{\overline{F}}_1 = X_4. \tag{16}$$

Further by differentiating (15) and by using (13) we obtain

$$\begin{aligned} \dot{X}_2 &= -2X_4\left[1 + \gamma\overline{F}_1\right], \\ \dot{X}_3 &= -mX_4, \\ \dot{X}_4 &= -X_2\overline{F}_1 + mX_3 + \left(\left(\overline{F}_1\right)^2 - \lambda^2\right)\left(1 + \gamma\overline{F}_1\right). \end{aligned} \tag{17}$$

It can be seen that the system (16) and (17) has the following first integrals:

$$\begin{aligned} X_3^2 + X_4^2 &= \left(\lambda^2 - \left(\overline{F}_1\right)^2\right)X_2, \\ X_3 &= -m\left(\overline{F}_1 + \lambda\right), \\ X_2 + 2\overline{F}_1 + \gamma\left(\overline{F}_1\right)^2 &= C = \text{const}. \end{aligned} \tag{18}$$

From system (16)–(18) we derive the following second order equation

$$\ddot{\overline{F}}_1\left(\lambda^2 - \left(\overline{F}_1\right)^2\right) + \left(\dot{\overline{F}}_1\right)^2\overline{F}_1 + \lambda m^2\left(\lambda + \overline{F}_1\right)^2 + \left(\lambda^2 - \left(\overline{F}_1\right)^2\right)^2\left(1 + \gamma\overline{F}_1\right) = 0. \tag{19}$$

The boundary conditions corresponding to (19) follow from (11, line 1) and the condition of global equilibrium of forces in the x_{10} direction. Thus, they read

$$\overline{F}_1(0) = -\lambda, \quad \overline{F}_1(1) = -\lambda. \tag{20}$$

Following Antman and Kenny (1981), we introduce the angle θ (the Euler angle of nutation) between $\mathbf{e}_1$ and $\mathbf{e}_{10}$. Then,

$$\overline{F}_1 = -\lambda\cos\theta. \tag{21}$$

With (21), Eq. (19) becomes

$$\lambda^3\sin^3\theta\left[\ddot{\theta} + m^2\frac{\sin\theta}{(1+\cos\theta)^2} + \lambda\sin\theta(1 - \gamma\lambda\cos\theta)\right] = 0. \tag{22}$$

We are going to use (22) for the *local* bifurcation analysis. Thus, we are interested in solutions $\theta(t)$ of (22) that are $C^2(0,1)$ functions, small in L_∞ norm. Therefore, as shown by Antman and Kenny (1981), we assume that $|\theta| < \pi$. Then, from (22) we obtain

$$\ddot{\theta} + m^2\frac{\sin\theta}{(1+\cos\theta)^2} + \lambda\sin\theta(1 - \gamma\lambda\cos\theta) = 0. \tag{23}$$

The boundary conditions corresponding to (23) follow from (20) and (21) so that

$$\theta(0) = 0, \quad \theta(1) = 0. \tag{24}$$

The system (23) and (24) will be the basis for our bifurcation analysis. Note that (23) and (24) are Euler–Langrange equations of the variational principle that we formulate next. Consider the problem of determining the minimum of the functional

$$I = \int_0^1 \left(\frac{\dot{\Theta}^2}{2} - \frac{m^2}{(1+\cos\Theta)} + \lambda\left[\cos\Theta + \gamma\frac{\lambda}{2}\sin^2\Theta\right]\right) \mathrm{d}t, \tag{25}$$

where $\Theta \in Y$ with

$$Y = \left\{\Theta : \Theta \in C^2(0,1), \quad \Theta(0) = \Theta(1) = 0\right\}. \tag{26}$$

Then on the solution of (23) the necessary condition for the minimum of I is satisfied, i.e.,

$$\delta I(\theta, w) = 0, \tag{27}$$

where $w = \Theta - \theta$.

From system (16) and (18) we can obtain an important relation that we shall use later. Namely, by substituting in (18, line 1) the variables X_3, X_2 and X_4 from (18, line 2), (18, line 3) and (16) respectively, we obtain

$$\left(\dot{\overline{F}}_1\right)^2 = \left(\lambda + \overline{F}_1\right)\left\{\left(\lambda - \overline{F}_1\right)\left(C - 2\overline{F}_1 - \gamma\left(\overline{F}_1\right)^2\right) - m^2\left(\lambda + \overline{F}_1\right)\right\}. \tag{28}$$

The boundary conditions corresponding to (28) are given by (20). We shall be concerned with the solutions of (28) that are symmetric with respect to the middle point $t = 1/2$. In this case we have $\dot{\overline{F}}_1(1/2) = 0$. Note also that (28) is a generalization of the result presented in Iljohin (1979) where the classical Bernoulli–Euler rod is treated. Our first integral reduces to the one presented in Iljohin (1979) when we set $\gamma = 0$ in (28).

3. Bifurcation analysis

The bifurcation analysis will be based on the *second* order nonlinear boundary value problem (23) and (24). First, we write (23) and (24) in a compact form. Thus, we define a nonlinear operator $\mathscr{F}(\Theta, m^2)$ mapping $Y \times R_+ \to Z$, where Y is given by (26) and $Z = C(0,1)$ is the space of continuous functions. Note that we shall treat λ and γ as fixed and given in advance while m^2 will be considered as a bifurcation parameter. With

$$\mathscr{F}\left(\Theta, m^2\right) = \ddot{\Theta} + m^2\frac{\sin\Theta}{(1+\cos\Theta)^2} + \lambda\sin\Theta(1 - \gamma\lambda\cos\Theta), \tag{29}$$

system (23) and (24) becomes

$$\mathscr{F}\left(\theta, m^2\right) = 0. \tag{30}$$

Note that $\theta_0 = 0$ satisfies (30) for *all* values of $m^2 > 0$. The first Fréchet derivative of $\mathscr{F}(\Theta, m^2)$ calculated on $\theta_0 = 0$ reads

$$D^{(1)}\mathscr{F}\left(\theta_0 = 0, m^2\right)w = B\left(m^2\right)w = \ddot{w} + w\left[\frac{m^2}{4} + \lambda(1 - \gamma\lambda)\right]. \tag{31}$$

The lowest eigenvalue m_1 and corresponding eigenfunction w_1 of the equation $B(m^2)w = 0$ are

$$\frac{m_1^2}{4} + \lambda(1 - \gamma\lambda) = \pi^2, \qquad w_1(t) = D \sin \pi t, \tag{32}$$

where D is an arbitrary constant. Without loss of generality we take $D = 1$. Our next goal is to show that $(0 \times m_1^2) \in Y \times R_+$ is a bifurcation point of $\mathscr{F}(\theta, m^2) = 0$. Note that $B(m^2)$ is a self-adjoint linear operator $B(m^2) = B^*(m^2)$. Therefore the eigenvector of $B^*(m^2)$ corresponding to m_1^2 is

$$q_1(t) = Q \sin \pi t, \tag{33}$$

where Q is an arbitrary constant that we shall, again, take as $Q = 1$. From (29) we conclude that

$$\mathscr{F}(\theta, m^2) = -\mathscr{F}(-\theta, m^2). \tag{34}$$

For $m_1^2 - \Delta m_1^2 \leqslant m^2 \leqslant m_1^2 + \Delta m_1^2$ with $|\Delta m_1^2|$ small, we assume the solution to (30) in the form

$$\theta = a \sin \pi t + u^*(a, \Delta m_1^2, t). \tag{35}$$

Eq. (34) implies that $u^*(a, \Delta m_1^2, t)$ is at least of the order $O(a^3)$ (see Golubitsky and Schaeffer, 1985; p. 300). The parameter a (the amplitude in the terminology of Troger and Steindl (1991)) is determined from the bifurcation equation

$$f(a, \Delta m_1^2) = \int_0^1 \left[\mathscr{F}\left(a \sin \pi t + u^*(a, \Delta m_1^2, t), m_1^2 + \Delta m_1^2\right)\right] q_1(t)\, dt. \tag{36}$$

Eq. (36) can be written as

$$f(a, \Delta m_1^2) = c_1 a \Delta m_1^2 + c_3 a^3 + c_5 a^5 + c_7 a^7 + h.o.t., \tag{37}$$

where $h.o.t.$ denotes terms of the order $O(a^9, (\Delta m_1^2)a^3)$ and

$$\begin{aligned}
c_1 &= \frac{1}{4} \int_0^1 w_1^2(t)\, dt = \frac{1}{8}, \\
c_3 &= -\frac{k}{6} \int_0^1 w_1^4(t)\, dt = -\frac{k}{16}, \\
c_5 &= \int_0^1 \left[-\frac{k}{12} w_1^3(t) U_{aaa}(t) + \frac{l}{240} w_1^6(t) \right] dt, \\
c_7 &= \int_0^1 \left\{ -k \left[\frac{1}{72} w_1^2(t) U_{aaa}^2(t) + \frac{1}{240} w_1^3(t) U_{aaaaa}(t) \right] + \frac{l}{288} w_1^5(t) U_{aaa}(t) + \frac{r}{5040} w_1^8(t) \right\} dt.
\end{aligned} \tag{38}$$

In (38) we used k, l and r to denote the following constants

$$\begin{aligned}
k &= 3\lambda - 6\gamma\lambda^2 - 2\pi^2, \\
l &= -15\lambda - 15\gamma\lambda^2 + 17\pi^2, \\
r &= -63\lambda + 126\gamma\lambda^2 + 62\pi^2
\end{aligned} \tag{39}$$

and the functions $U_{aaa}(t)$ and $U_{aaaaa}(t)$ are solutions of the linear system of differential equations (see Golubitsky and Schaeffer, 1985; p. 33)

$$\begin{aligned}
&B(m_1^2) U_{aaa} + ED^{(3)}\mathscr{F}(w_1, w_1, w_1) = 0, \\
&B(m_1^2) U_{aaaaa} + 10ED^{(3)}\mathscr{F}(w_1, w_1, U_{aaa}) + ED^{(5)}\mathscr{F}(w_1, w_1, w_1, w_1, w_1) = 0.
\end{aligned} \tag{40}$$

In (40) we used $D^{(3)}\mathscr{F}$ and $D^{(5)}\mathscr{F}$ to denote the third and fifth Fréchet derivative of $\mathscr{F}(\Theta, m^2)$ with respect to Θ, calculated at the point $(\theta_0 = 0, m_1^2)$. Those derivatives are calculated in the directions

$(w_1, w_1, w_1), (w_1, w_1, U_{aaa})$ and $(w_1, w_1, w_1, w_1, w_1)$, respectively. Also in (40) we used E to denote the projection operator mapping Z onto the range of $B(m_1^2)$. With $\mathscr{F}$ and B given by (29) and (31) we obtain

$$\begin{aligned} \ddot{U}_{aaa} + \pi^2 U_{aaa} &= -\frac{k}{4}\sin 3\pi t, \\ \ddot{U}_{aaaaa} + \pi^2 U_{aaaaa} &= \frac{5}{32}\left\{\frac{k^2}{\pi^2} + l\right\}\sin 3\pi t - \frac{1}{64}\left\{\frac{5k^2}{\pi^2} + 2l\right\}\sin 5\pi t. \end{aligned} \tag{41}$$

Note that (see Golubitsky and Schaeffer, 1985; p. 32)

$$\begin{aligned} U_{aaa} &= \left[\partial^3 u^*\left(a, \Delta m_1^2, t\right)/\partial a^3\right]_{a=\Delta m_1^2=0}, \\ U_{aaaaa} &= \left[\partial^5 u^*\left(a, \Delta m_1^2, t\right)/\partial a^5\right]_{a=\Delta m_1^2=0}. \end{aligned} \tag{42}$$

Keeping in mind that u^* does not belong to the null space of $B(m_1^2)$ the solutions to (41) are

$$\begin{aligned} U_{aaa} &= \frac{k}{32\pi^2}\sin 3\pi t, \\ U_{aaaaa} &= -\frac{5}{256\pi^2}\left\{\frac{k^2}{\pi^2} + l\right\}\sin 3\pi t + \frac{1}{1536\pi^2}\left\{\frac{5k^2}{\pi^2} + 2l\right\}\sin 5\pi t. \end{aligned} \tag{43}$$

In writing (41, line 2) we used the solution of (41, line 1) given by (43, line 1). By using (43) in (38, lines 3 and 4) we find

$$c_5 = \frac{k^2 + 4l\pi^2}{3072\pi^2}, \quad c_7 = \frac{4r\pi^4 - k^3 - 2kl\pi^2}{73728\pi^4}. \tag{44}$$

For the qualitative study of solutions, Eq. (37) may be simplified by neglecting *h.o.t.* First note that $c_1 > 0$ for all values of the parameters. Then, we distinguish the following special cases:

1. Suppose that $k \neq 0$. Then, from (38, line 2), we have $c_3 \neq 0$ and (37) is contact equivalent (see Keyfitz, 1986) to

   $$a\Delta m_1^2 - \text{sgn}(k)a^3 = 0. \tag{45}$$

   From (39, line 1) we conclude that k as a function of λ has two zeros, that separate sub- from super-critical bifurcations (see Atanackovic and Glavardanov, 1996).
2. Suppose that, $k = 0$ and $l \neq 0$. Then, $c_3 = 0$, and $c_5 \neq 0$, so that (37) is contact equivalent to

   $$a\Delta m_1^2 + \text{sgn}(l)a^5 = 0. \tag{46}$$

   Again the bifurcation can be both sub- and super-critical. There exist a single point λ^* where l as a function of λ vanishes (remember that $k = 0$)

   $$\lambda^* = \frac{44}{45}\pi^2. \tag{47}$$

   This point separates the sub- from super-critical bifurcation. For $\lambda < \lambda^*$ we have the sub-critical bifurcation while for $\lambda > \lambda^*$ the bifurcation is super-critical.
3. Suppose that $k = l = 0$. Thus, $c_3 = c_5 = 0$ and

   $$c_7 = \frac{5}{4608}\pi^2. \tag{48}$$

   Therefore (37) is contact equivalent to

   $$a\Delta m_1^2 + a^7 = 0, \tag{49}$$

   i.e., the bifurcation is sub-critical.

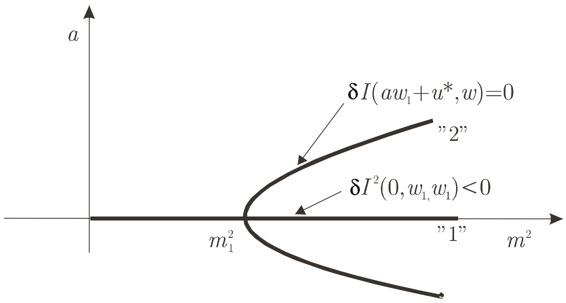

Fig. 3. Bifurcation pattern.

Next we examine the stability of the trivial configuration for $m^2 > m_1^2$. We shall use the functional (25). Consider the trivial branch $a = 0$ in Fig. 3. The second variation of I calculated on the trivial solution $\theta_0 = 0$ reads

$$\delta^2 I(\Theta = 0, w, w) = \int_0^1 \left[\dot{w}^2 - m^2 w^2 - \lambda(1 - \gamma\lambda)w^2\right] \mathrm{d}t. \tag{50}$$

Note that the variation w satisfies $w(0) = w(1) = 0$. Suppose that $m^2 = m_1^2 + \Delta m_1^2$. Then, we have

$$\int_0^1 \left[\dot{w}_1^2 - m_1^2 w_1^2 - \lambda(1 - \gamma\lambda)w_1^2\right] \mathrm{d}t = 0. \tag{51}$$

We calculate the second variation Eq. (50) in the direction w_1 and for $m^2 = m_1^2 + \Delta m_1^2$. The result is

$$\delta^2 I(\theta_0 = 0, w_1, w_1) = -\Delta m_1^2 \int_0^1 w_1^2(t)\, \mathrm{d}t < 0, \tag{52}$$

where we used Eq. (51). The inequality (52) shows that on the branch '1' in Fig. 3 the functional I is *not* in minimum since it does not satisfy the necessary condition for the minimum $\delta^2 I(\Theta = 0, w, w) > 0$ for *all* w. Therefore, according to the energy stability criteria for $m^2 > m_1^2$ the trivial branch is not stable.

Note that the integrand in functional (25) does not depend on t. Therefore, there is Jacobi type first integral of the Euler–Lagrange equations (23) that reads (see Vujanovic and Jones, 1989)

$$\frac{\dot{\theta}^2}{2} + \frac{m^2}{(1 + \cos\theta)} - \lambda\left[\cos\theta + \gamma\frac{\lambda}{2}\sin^2\theta\right] = \text{const.} \tag{53}$$

The integral (53) could be obtained from (28) by using the relation (21).

4. Solution of Eq. (28)

With $G = \overline{F}_1$, $\overline{F}_1(1/2) = G_1$ and by using $\dot{\overline{F}}_1(1/2) = 0$, to determine C, we obtain from (28)

$$C = \gamma G_1^2 + 2G_1 + \frac{m^2(\lambda + G_1)}{\lambda - G_1}. \tag{54}$$

Since $-\lambda$ and G_1 are zeros of the right hand side of (28) it follows that (28) may be written as

$$\dot{G}^2 = (\lambda + G)(G - G_1)\left\{\gamma G^2 + (2 + \gamma G_1 - \gamma\lambda)G - \lambda\left(2 + \gamma G_1 + \frac{2m^2}{\lambda - G_1}\right)\right\}. \tag{55}$$

From (55) we obtain

$$\dot{G}^2 = \gamma(\lambda + G)(G - G_1)(G - b)(G - c), \tag{56}$$

where

$$b = \frac{-(2 + \gamma G_1 - \gamma\lambda) - \sqrt{(2 + \gamma G_1 + \gamma\lambda)^2 + 8\gamma\lambda\frac{m^2}{\lambda - G_1}}}{2\gamma},$$
$$c = \frac{-(2 + \gamma G_1 - \gamma\lambda) + \sqrt{(2 + \gamma G_1 + \gamma\lambda)^2 + 8\gamma\lambda\frac{m^2}{\lambda - G_1}}}{2\gamma}. \tag{57}$$

Note that $t = 0$ corresponds to $G = -\lambda$ and $t = 1/2$ corresponds to $G = G_1$. Therefore if there is a solution to (56) we must have $\gamma(\lambda + G)(G - G_1)(G - b)\ (G - c) > 0$ when $G \in (-\lambda, G_1)$. In what follows we assume that the parameters in (56) and (57) are chosen so that this condition holds.

We note that for the case of a rod without axis compressibility and with neglected influence of shear stresses, i.e., $\gamma = 0$ the polynomial on the right hand side of (28) is of the third order. The integration procedure in this case may follow the method used for Lagrange top (see Whittaker, 1965).

By separating variables in (56) we obtain

$$\frac{1}{2} = \frac{2}{\sqrt{\gamma(\lambda + c)(G_1 - b)}} F\left(\frac{\pi}{2}, q\right), \tag{58}$$

where $F(\phi, q) = \int_0^\phi \mathrm{d}x \Big/ \sqrt{1 - q^2 \sin^2 x}$ and $q = \sqrt{((\lambda + G_1)(c - b))/((\lambda + c)(G_1 - b))}$. Note that when $\theta \to 0$ we have

$$\frac{m^2}{4} + \lambda(1 - \gamma\lambda) = \pi^2 \tag{59}$$

in agreement with (32). Thus the solution (58) confirms the bifurcation analysis presented in previous section.

Eq. (58) can be used to calculate G_1 if m, λ and γ are prescribed. We shall compare the so determined G_1 with the value obtained from numerical integration of the system (10) and (11).

5. Numerical results

In this section we present numerical solutions of the system (10) and (11) for specific values of parameters λ, m, α, A_{22}/A_{11} and β. Note that, in engineering terms, A_{22}/A_{11} represents the ratio between bending (A_{22}) and torsional (A_{11}) stiffness. As stated earlier (see the comment after Eq. (11)) the system (10) and (11) has rotational symmetry so that, without loss of generality, we shall choose a solution that satisfies $\overline{M}_3(0) = 0$. Once the solution to (10) and (11) is obtained we compare $\overline{F}_1(1/2)$ with G_1 determined from (58) with given λ, m and $\gamma = \alpha - \beta$.

In Fig. 4(a) and (b) we show the projections of the rod axis on the $\bar{x}_{10} - \bar{x}_{20}$ and $\bar{x}_{10} - \bar{x}_{30}$ coordinate planes, respectively. The shape is determined for $\lambda = 9, A_{22}/A_{11} = 1.3$ and $m = 2$ and for four different choices of α and β. The curve 1 corresponds to the Bernoulli–Euler rod, i.e., ($\alpha = 0, \beta = 0$), the curve 2 corresponds to ($\alpha = 0.00111, \beta = 0$), the curve 3 corresponds to ($\alpha = 0, \beta = 0.00288$) and finally the curve 4 corresponds to ($\alpha = 0.00111, \beta = 0.00288$).

In Table 1 we present the values of $\overline{F}_1(1/2), \overline{M}_2(0)$ and the constant C in the first integral (18, line 3) for different values of α and β.

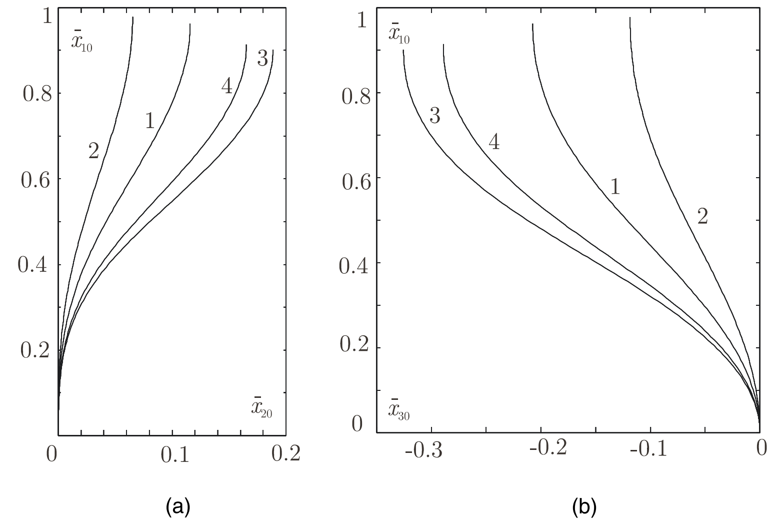

Fig. 4. Post-critical shape of the rod.

Table 1
The values at the boundary and the value of the constant in the first integral for different values of parameters

α	β	$\overline{F}_1(1/2)$	$\overline{M}_2(0)$	C
0	0	−8.32746	1.22488	−16.49966
0.00111	0	−8.78083	0.69520	−17.42669
0	0.00288	−7.32887	1.95715	−14.40354
0.00111	0.00288	−7.68626	1.72666	−15.16263

Note that, because of the global equilibrium, we have $F_2(0) = F_3(0) = 0$ so that the only unknown dependent variable at $t = 0$ is $\overline{M}_2(0)$. In the process of numerical integration of the system (10) and (11) we calculated the first integrals (14) at each step of integration. They were constant up to terms of the order 10^{-8}.

6. Conclusion

In this paper we treated the problem of determining the stability boundary and post-critical behavior of an elastic rod loaded by a force and a couple at its ends. Our main results may be stated as:

1. By a suitable transformation of dependent variables, we reduced the initial system of equilibrium equations (18 of them) to a single second order nonlinear differential Eq. (23). The linearization of this equation has a null space of dimension 1. In previous works (see Béda et al., 1992; Atanackovic and Glavardanov, 1996) the equilibrium equations were reduced to two second order differential equations that lead to null space of dimension 2.

The bifurcation analysis was applied to Eq. (23). We concluded that there is the possibility of both super- and sub-critical bifurcation. Thus, we confirmed and extended our previous results presented in Atanackovic and Glavardanov (1996). Namely, here by calculating the higher order terms in the bifurcation equation, we determined the type of bifurcation for *all* values of the parameters m, λ, α and β.

2. For Eq. (23) we formulated a variational principle (27). We used this principle to show that the trivial branch becomes unstable (the energy type of functional (25) is *not* a minimum) when the bifurcation parameter passes the critical value.

3. The first integral (18, line 3) was transformed into the form (28). This integral could also be obtained as a Jacobi-type first integral (see Vujanovic and Jones, 1989) of the variational principle (27). By using (28) we were able to express $\overline{F}_1(t)$ in terms of elliptic integrals (see (58)). The value $\overline{F}_1(1/2)$ calculated from elliptic integrals representation was used to check the numerical integration.

4. Finally, we solved the system (10) and (11) numerically. The results, i.e., the post-critical shape of the rod are shown in Fig. 4 for several values of parameters m, λ, α, A_{22}/A_{11} and β. From this figure we conclude that for fixed m, λ and A_{22}/A_{11} the maximal deflection increases with increasing β and decreases with increasing α.

Acknowledgements

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